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A WAVE EQUATION FOR RADIATING SOURCE DISTRIBUTIONS. (U)

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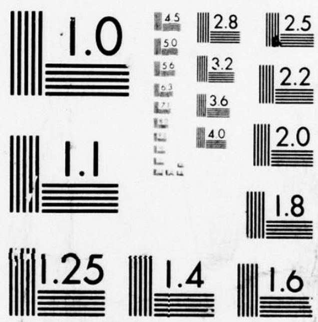
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A WAVE EQUATION FOR RADIATING SOURCE DISTRIBUTIONS

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ABSTRACT

The Bleistein-Cohen separation of a source (for the inhomogeneous Helmholtz wave equation) distribution into radiating and non-radiating portions is reformulated into a form suitable for deriving a wave equation governing the radiating portion of the source distribution.

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SECTION I

INTRODUCTION

Bleistein and Cohen have shown [1,2] that a source distribution, giving rise to fields that satisfy the inhomogeneous Helmholtz wave equation (for acoustics, electromagnetics, etc.), can be separated into radiating and non-radiating portions; and that the radiating source distribution can be determined completely and uniquely in a well-conditioned manner from knowledge of the radiated fields on any closed surface exterior to the support of the source distribution.

An alternative formulation of this separation is developed in a form suitable for deriving a wave equation which governs the radiating portion of the source distribution.

SECTION II

AN ALTERNATIVE FORMULATION OF THE BLEISTEIN-COHEN EQUATIONS

Let $Y_{mn}(\theta,\phi)$ and $j_l(kr)$ be the normalized over $\theta \in (0,\pi)$ and $\phi \in (0,2\pi)$ spherical harmonic and the normalized over $r \in (0,\infty)$ spherical Bessel functions respectively. It thus follows that the functions $Y_{mn}(\theta,\phi)\ j_l(kr)$ are orthonormal and complete in all space [3], and that any function $\rho(\theta,\phi,r)$ can be expanded as

$$\rho(\theta,\phi,r) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \sum_{l=0}^{\infty} C_{mnl}\, Y_{mn}(\theta,\phi)\ j_l(kr) \quad , \tag{1}$$

where the coefficients C_{mnl} are given by

$$C_{mnl} = \int_0^{\infty}\int_0^{2\pi}\int_0^{\pi} Y_{mn}(\theta,\phi)\ j_l(kr)\ \rho(\theta,\phi,r)\ \sin\theta\ d\theta\ d\phi\ r^2\ dr \quad . \tag{2}$$

If the function ρ represents a source distribution, giving rise to fields ψ, which satisfy the inhomogeneous reduced Helmholtz wave equation

$$\nabla^2\psi + k^2\ \psi = -\rho \quad , \tag{3}$$

and the support of these sources ρ is confined to the interior of a sphere of radius a, then the radiated fields ψ at the exterior of this sphere are

$$\psi(\theta,\phi,r) = ik \sum_{n=0}^{\infty} \sum_{m=-n}^{n} Y^{*}_{mn}(\theta,\phi)\ h^{(1)}_n(kr) \int_0^{a}\int_0^{2\pi}\int_0^{\pi} Y_{mn}(\theta',\phi')\ j_n(kr')$$

$$\times\ \rho(\theta',\phi',r')\ \sin\theta'\ d\theta'\ d\phi'\ r'^2\ dr' \quad , \quad r > a \quad , \tag{4}$$

which by (2) can be written as

$$\psi(\theta,\phi,r) = ik \sum_{n=0}^{\infty} \sum_{m=-n}^{n} C_{mnn} Y^{*}_{mn}(\theta,\phi)\, h^{(1)}_{n}(kr) \quad ; \tag{5}$$

i.e., only the diagonal coefficients C_{mnn} (of the complete set of coefficients C_{mnl}) of the expansion (1,2) give rise to radiated fields.

If the functions G_{mnl} are defined as

$$G_{mnl} \equiv Y_{mn}(\theta,\phi)\, j_{l}(kr) \quad , \tag{6}$$

then by the general solution in spherical coordinates of the homogeneous reduced Helmholtz wave equation,

$$\nabla^2 G_{mnn} + k^2 G_{mnn} = 0 \quad . \tag{7}$$

Thus, by Green's theorem, (3), and (2)

$$\begin{aligned}\oint_S d\mathbf{s}\cdot(G_{mnn}\nabla\psi - \psi\nabla G_{mnn}) &= \int_V dv\,(G_{mnn}\nabla^2\psi - \psi\nabla^2 G_{mnn}) \\ &= \int_V dv\, G_{mnn}\,\rho \\ &= C_{mnn} \quad ;\end{aligned} \tag{8}$$

i.e., the diagonal coefficients, which completely and uniquely determine the radiating portion of the source distribution, can be determined in a

well-conditioned manner from knowledge of the field ψ on any closed surface S, exterior to the support of the source distribution ρ.

SECTION III

A WAVE EQUATION FOR RADIATING SOURCE DISTRIBUTIONS

The expansion (1) of an arbitrary source distribution $\rho(\theta,\phi,r)$ can be broken into the diagonal and non-diagonal parts

$$\rho(\theta,\phi,r) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \sum_{\substack{l=0 \\ l\neq n}}^{\infty} C_{mnl}\, Y_{mn}(\theta,\phi)\, j_l(kr)$$

$$+ \sum_{n=0}^{\infty} \sum_{m=-n}^{n} C_{mnn}\, Y_{mn}(\theta,\phi)\, j_n(kr) \quad , \tag{9}$$

which can be identified respectively as the non-radiating ($\rho_\circ$) and radiating (ρ_∞) portion of the source distribution; i.e.,

$$\rho(\theta,\phi,r) = \rho_\circ(\theta,\phi,r) + \rho_\infty(\theta,\phi,r) \quad , \tag{10}$$

where

$$\rho_\circ(\theta,\phi,r) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \sum_{\substack{l=0 \\ l\neq n}}^{\infty} C_{mnl}\, Y_{mn}(\theta,\phi)\, j_l(kr) \quad , \tag{11}$$

$$\rho_\infty(\theta,\phi,r) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} C_{mnn}\, Y_{mn}(\theta,\phi)\, j_n(kr) \quad . \tag{12}$$

By the general solution in spherical coordinates of the homogeneous reduced Helmholtz wave equation, it follows from (12) that

$$\nabla^2\rho_\infty + k^2\,\rho_\infty = 0 \quad . \tag{13}$$

An alternative, and much simpler, coordinate independent derivation of (13) for the far-field radiating source distribution can be accomplished in the following fashion.

The three-dimensional (spatial) Fourier transform of any source distribution

$$\tilde{\rho}(\mathbf{k},\omega) = \int e^{i\mathbf{k}\cdot\mathbf{x}}\,\rho(\mathbf{x},\omega)\,d^3x \quad , \tag{14}$$

the support of which, in general, is infinite in k-space, can always be separated into the two functions $\tilde{\rho}_o(\mathbf{k},\omega)$ and $\tilde{\rho}_\infty(\mathbf{k},\omega)$, the support of which is off and on the surface of the k-space sphere of radius $k=\frac{\omega}{c}$ respectively; i.e.,

$$\tilde{\rho}_o(\mathbf{k},\omega) \equiv \begin{cases} 0 & , \quad \forall\, k = \frac{\omega}{c} \,, \\ \tilde{\rho}(\mathbf{k},\omega) & , \quad \forall\, k \neq \frac{\omega}{c} \,, \end{cases} \tag{15}$$

$$\tilde{\rho}_\infty(\mathbf{k},\omega) \equiv \begin{cases} \tilde{\rho}(\mathbf{k},\omega) & , \quad \forall\, k = \frac{\omega}{c} \,, \\ 0 & , \quad \forall\, k \neq \frac{\omega}{c} \,, \end{cases} \tag{16}$$

$$\tilde{\rho}(\mathbf{k},\omega) = \tilde{\rho}_o(\mathbf{k},\omega) + \tilde{\rho}_\infty(\mathbf{k},\omega) \quad . \tag{17}$$

However, by (16), $\tilde{\rho}_\infty(\mathbf{k},\omega)$ is the complete set of the (range and phase normalized) radiated far-field of the source distribution $\rho(\mathbf{x},\omega)$, and by (15), $\tilde{\rho}_o(\mathbf{k},\omega)$ contributes nothing to the radiated far-field. it thus follows that

$$\rho_o(\mathbf{x},\omega) = \frac{1}{(2\pi)^3}\int_{\forall\, k\neq\frac{\omega}{c}} e^{-i\mathbf{k}\cdot\mathbf{x}}\, \tilde{\rho}_o(\mathbf{k},\omega)\, d^3k \tag{18}$$

is the non far-field radiating portion of the source distribution, and

$$\rho_\infty(\mathbf{x},\omega) = \frac{1}{(2\pi)^3}\int_{\forall\, k=\frac{\omega}{c}} e^{-i\mathbf{k}\cdot\mathbf{x}}\, \tilde{\rho}_\infty(\mathbf{k},\omega)\, d^3k \tag{19}$$

is the far-field radiating portion of the source distribution; where

$$\rho(\mathbf{x},\omega) = \rho_o(\mathbf{x},\omega) + \rho_\infty(\mathbf{x},\omega) \quad . \tag{20}$$

It follows directly from (19) that

$$\nabla^2\rho_\infty + k^2\,\rho_\infty = 0 \quad , \quad k = \frac{\omega}{c} \quad . \tag{21}$$

As suggested by Bleistein [4], expansion of the integrand term $e^{-i\mathbf{k}\cdot\mathbf{x}}$ in (14), (18), and (19) in spherical harmonics reveals that the radiating and non-radiating source distributions of this alternative derivation are indeed the same radiating and non-radiating source distributions given by (11) and (12) respectively.

It can similarly be shown that a vector current density distribution $\mathbf{J}$, which gives rise to electromagnetic fields that satisfy Maxwell's equations, can similarly be separated into radiating and non-radiating current density distributions, and that the radiating current density distribution $\mathbf{J}_\infty$ satisfies the vector wave equation

$$\nabla\times\nabla\times\mathbf{J}_\infty - k^2\,\mathbf{J}_\infty = 0 \quad . \tag{22}$$

SECTION IV

CONCLUDING REMARKS

Source distributions that fail to satisfy the wave equation should be carefully excluded from solutions to the inverse source, inverse scattering, and synthesis problem. Such an exclusion might advantageously be achieved by viewing this wave equation as an additional necessary condition (e.g., facilitating regularization of ill-conditioned and/or ill-posed problems) on the solutions to the inverse source, inverse scattering, and synthesis problem.

SECTION V

REFERENCES

[1] Bleistein, N., and Cohen, J. K., *Non-uniqueness in the inverse source problem in acoustics and electromagnetics*, J. Math. Physics, 18, 2, 194-201 (1977).

[2] Bleistein, N., and Cohen, J. K., *A note on the inverse source problem*, J. Math. Physics, (to be published).

[3] Watson, G. N., *A Treatise on the Theory of Bessel Functions*, Sect. 16.4., p. 533, Cambridge University Press, 1962.

[4] Bleistein, N., Personal communication.

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